REPRINT OF OLD QUESTIONS.

15 [*1914*, 278; *1916*, 353; *1919*, 68; *1920*, 114, 361; *1921*, 124]. In the *Proceedings of the Royal Society of Edinburgh*, vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

Students are notoriously suspicious of those steps which an author announces as "easy," and are sometimes inclined to believe that the word is used in a humorous sense. The present case almost justifies such an attitude, since so far no reader has furnished a satisfactory explanation, and it is difficult to resist the suspicion that the "proof" is an illusion.

In a previous editorial note (*1920*, 361) a statement was made of the principal types of answer arriving in response to this question. To prevent the repetition of a former misunderstanding, it may be said here that the question is not how the second equation may be deduced from the first, but how the second equation may be used to facilitate the proof that the first is impossible. There are of course proofs in existence that the sum of two cubes cannot be a cube.[1]

21 [*1914*, 341; *1916*, 354; *1919*, 68, 239; *1920*, 114; *1921*, 114]. For the Diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$\begin{array}{lrrrrrrrr} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given Diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

A note by E. B. Escott (*1919*, 239) contained several methods for finding solutions, and references to the literature[2] suggested the probability that an infinite number of integral solutions existed; but no others were given. Hence both parts of the question await a final reply.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460; *1921*, 19, 125]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

Although nine replies by six different writers have been published, the problem is by no means exhausted. As to the interpretation of the question, there is no problem worth discussing if h is taken to be constant, so that each correspondent has naturally assumed that the equation is to be true for a variable h over the interval $0 < h < H$. Starting from the well-known fact that the equation is true for polynomials of degree three at most,[3] E. Swift (*1920*, 301) proved that it was

[1] See, for instance, R. D. Carmichael, *Diophantine Analysis*, p. 67.

[2] L. J. Mordell, *Proceedings of London Mathematical Society*, series 2, vol. 13, 1914, pp. 60–80. See also L. E. Dickson, *History of the Theory of Numbers*, vol. 2, pp. 537–9.

[3] The prismatoid formula is that special case of Cotes' formula of approximate integration for which the interval is divided into *two* equal parts, and the integrand assumed to be of degree two. When the method is applied to a polynomial of degree three, the interpolation for the

true for no other analytic functions regular at the origin, and A. A. Bennett (*1920*, 301) obtained a somewhat more comprehensive result. D. C. Gillespie (*1920*, 405) showed that the equation was satisfied by no other function having six continuous derivatives in the interval. On the other hand, A. A. Bennett (*1920*, 460) found analytic solutions of the type $f(x) = x^2 \sin(b \log x - c)$ having essential singularities at the origin; and, in editorial remarks by W. A. Hurwitz (*1920*, 462), it was shown (1) that the value of b may be real, (2) that the function then has, from the real variable point of view, a continuous derivative at the origin. The main question now is: *How many continuous derivatives at most may $f(x)$ possess at the origin, if $f(x)$ satisfies the given equation, but is not of the form $ax^3 + bx^2 + cx + d$?* The answer may be anything from *one* to *five* inclusive.

Throughout the discussion the behavior of the function in the neighborhood of the origin is the point of critical importance, and the problem is fundamentally different when so modified that the identity of the origin is lost. Thus the related question where the equation is to be true after an arbitrary translation of axes was satisfactorily answered by J. P. Ballantine (*1921*, 19), who showed that $ax^3 + bx^2 + cx + d$ was the only *continuous* solution.[1]

36 [*1919*, 69, 291–295], part 1. For what values of n can $\cos 2\pi/n$ be expressed in the form $(a + \sqrt{b})/c$, where a, b, and c are integers?

The other parts of this question, relating to equations of the third and fourth degrees, have been answered (*1919*, 292). In a recent issue (*1921*, 374) R. S. Underwood answered a question very near to this, showing that, of angles commensurable with π, only multiples of $\pi/3$ have *rational* cosines. We know that the answer to the present question includes multiples of $\pi/4$, $\pi/5$ and $\pi/6$. What else does it include?

39 [*1920*, 256; *1921*, 125]. There are certain problems in geometry which are simple in statement but can be reduced only to very complicated problems in transcendental analysis. Following are several examples of the type of problem in question.

1. What is the smallest plane area within which a given figure can be turned through a complete revolution? It is not implied that the figure should revolve about a fixed point, but merely that in the course of its motion it should have every possible orientation in the plane. The problem may be modified by considering only convex areas.

An interesting special case is that in which the given figure is a segment of a straight line. In this case it has been conjectured by Professors Osgood and Kubota that the smallest area may

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integrand is inexact, but the value of the definite integral is by accident correct. More generally, if $2n$ equal subdivisions are used, the definite integral will be found to be correctly evaluated, not only for an integrand of degree $2n$, but also for one of degree $2n + 1$. Gauss was perhaps the first to notice this fact (see *Encyclopédie des Sciences Mathématiques*, I 21, pp. 120–2). It is ignored in a number of the standard texts dealing with interpolation; though it is quite evident when central differences are used. In fact, when the interpolation is made from *any* $2n + 1$ values of x, as $x_1, x_2, \ldots$, situated *symmetrically about* $h/2$, the error in approximating to a polynomial of degree $2n + 1$ is of the form $C(x - x_1)(x - x_2) \ldots$, which is an odd function of $x - h/2$, so that its integral between 0 and h vanishes.

[1] For further illustration of the importance of the fixed origin in the problem as proposed, the reader may contrast it with the case of the equation

$$\int_{x=-h/2}^{x=h/2} f(x)dx = (h/6)[f(-h/2) + 4f(0) + f(h/2)]$$

for which the most general continuous solution is $f(x) = a + bx^2 +$ (an odd continuous function).

be bounded by a three-cusped hypocycloid; if we consider only convex areas, perhaps the result will be an equilateral triangle. I have no indication of a proof.

2. For every closed convex curve of area P there is an n-sided circumscribed polygon of least area Q and an inscribed polygon of greatest area R. For a fixed value of the integer n and for all convex curves, what is the upper limit of Q/P and what is the lower limit of R/P? I have succeeded only in proving that for the case $n = 3$ the upper limit of Q/P is 2.

3. Let the area of a simple closed curve A be a. Remove from A the greatest possible area a_1 similar to another simple closed curve B. From the remaining figure remove the greatest possible area a_2 similar to B. Continue this process indefinitely. Is it or is it not true that

$$a_1 + a_2 + a_3 + \ldots = a?$$

I have proved the statement to be true in the special case that A is convex and B is a circle.

4. Let a given closed convex curve K have the property that a given triangle whose angles are incommensurable with π can be revolved completely within K (see part 1 of this question), always remaining inscribed to K. What may the curve K be? Can any other curve except a circle satisfy the conditions?

The questions contained in no. 39 were proposed by Professor S. Kakeya, and were contributed to this department at the request of the editors. The questions have aroused considerable interest, but up to the present have not been solved.[1]

41 [*1920*, 365; *1921*, 126]. A reader asks for an elementary proof of the following two propositions in number theory, either of which can readily be obtained from the other:

Every positive integer of the form $8n + 3$ *is the sum of three odd squares.*

Every positive integer is the sum of not more than three triangular numbers.

According to Bachmann[2] these results have only been proved by means of the theory of ternary forms.

42 [*1921*, 65, 126]. In connection with the questions of Kakeya, Professor W. B. Ford is led to the following inquiry: A line-segment AB is to be moved in its plane to a new position $A'B'$. How should this be done in order that the area generated may, to the greatest extent possible, be passed over three times?

Professor Ford has proved that, if the generated area is to be passed over, to the greatest possible extent, but *two* times, AB should be moved so that its instantaneous center of rotation is always on its right bisector.[3]

43 [*1921*, 260]. Is any rapid method known for the evaluation of the Sylvester determinant met with so often in elimination by the dialytic method? It would seem that there must be, both on account of its interesting shape, and of its frequent occurrence.

45 [*1921*, 305]. Is every non-trivial solution in integers of the equation $t^3 = x^3 + y^3 + 1$ expressible in the form $x = 9r^4 - 3r$, $y = 9r^3 - 1$, $t = 9r^4$? If there are non-trivial solutions not expressible in this form, can a general solution be found?

This problem arose out of a note by H. C. Bradley (*1921*, 307) in which he obtained the solution quoted by specialization of the formula

$$x = -(a - 3b)(a^2 + 3b^2) + 1, \qquad y = (a + 3b)(a^2 + 3b^2) - 1,$$
$$u = -(a^2 + 3b^2)^2 + (a + 3b), \qquad v = (a^2 + 3b^2)^2 - (a - 3b),$$

which was shown by Euler and Binet to be (except for a common multiplier) the most general rational solution of the equation $x^3 + y^3 = u^3 + v^3$, a and b being any rational numbers.[4] This leads at once to the general *rational* solution of the

[1] See, however, Question **42**, reprinted in this number.

[2] *Niedere Zahlentheorie*, Leipzig and Berlin, 1910, Teil 2, p. 325. See also Dickson, *History*, vol. 2, Chapter 7.

[3] W. B. Ford, "On Kakeya's minimum area problem," *Bulletin of the Amer. Math. Soc.*, vol. 28., 1922, pp. 45-53. Other references are given in this paper.

The editorial note in this MONTHLY (*1921*, 126) quotes Professor Ford's conclusions inaccurately.

[4] Carmichael, *Diophantine Analysis*, p. 65; Dickson, *History*, vol. 2, p. 555.

more special equation, so that the problem awaiting solution is one of *integral* values only.

DISCUSSIONS.

The following note is a reply to that of Professor E. T. Bell (*1920*, 413), in which the use of mathematical induction in elementary teaching was criticized on logical grounds. Professor Bell's position was that the method required the postulation of a special logical principle, and was therefore unsuited to beginners.

On Proofs by Mathematical Induction.

By R. S. Hoar, South Milwaukee, Wis.

The objections to the usual method of stating mathematical induction, raised by Piofessor Bell in the Monthly of November, 1920, seem to call for some further discussion.

When I was a boy, the part that confused *me* was: "We first establish a theorem for $n = 1$; then we show that, if it is true for $n - 1$, it is true for n." But, if n is 1, $n - 1$ must be zero. We have not proved it for zero, and we already know that it is true for n. Of course, such an attitude is absurd, even though natural. But is it absurd to object that we never prove the proposition for $n - 1$ equals anything (but merely for n equals something), and therefore never lay the foundation for the second step?

Why not state the process as follows? "We first show that, if the proposition is true for n, it is true for $n + 1$; then we show that it is true for a particular value of n, namely $n = 1$; then, since any whole number can be reached from any preceding whole number by successively adding 1 a finite number of times, the proposition is true for n equals any whole number."

The false proof that any assemblage contains an infinity of members, cited by Prof. Bell, is not based on mathematical induction in the form objected to by him, but rather is based upon the layman's substitute for mathematical induction; namely, that if a proposition is true for $n = 1$, $n = 2$, $n \doteq 3$, etc., repeated a "reasonable" number of times, then it must be true for any value of n.

The fact that such "proofs" are current among freshmen is probably due to the failure of their teachers to emphasize the necessity of taking the abstract general step from n to $n + 1$; just as the currency of "proofs" that two equals one, is due to the failure of teachers to introduce non-divisibility by zero at the very start, as a fundamental part of the concept of division, instead of dragging it in later as a rather lame exception, in order to defend division from the onslaughts of two-equals-one.

A real example of the non-applicability of mathematical induction would be the series set forth on page 15 of Huntington's *The Continuum* (Cambridge, Mass., 1917):

$$1_1\ 2_1\ 3_1\ \cdots; \qquad 1_2\ 2_2\ 3_2\ \cdots; \qquad \cdots.$$

Perhaps it would be well for teachers to exhibit both this series and the "infinite assemblage," in order that the students may realize the necessity for all three steps in mathematical induction.